

# Approximate solutions of Volterra-Fredholm integro-differential equations of fractional order

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## Abstract

In this study, sinc-collocation method is introduced for solving Volterra-Fredholm integro-differential equations of fractional order. Fractional derivative is described in the Caputo sense. Obtained results are given to literature as a new theorem. Some numerical examples are presented to demonstrate the theoretical results.

2010 Mathematics Subject Classification. **45J05**. 65L60; 26A33

Keywords. Volterra-Fredholm integro-differential equation, sinc-collocation method, Caputo derivative.

## 1 Introduction

Many problems, in science and engineering such as earthquake engineering, biomedical engineering, fluid mechanics can be modeled by fractional integro-differential equations. In order to better analysis these systems, it is required to obtain the solution of these equations. But, achieving the analytical solution of these equations can not be possible. Therefore, finding more accurate solutions using numerical schemes can be helpful. Some numerical algorithm for solving integro-differential equation of fractional order can be summarized as follows: but not limited to; Adomian decomposition method [16, 18, 19], Laplace decomposition method [32], Taylor expansion method [9], least squares method [17] differential transform method [5, 21], Spectral collocation method [14], Legendre wavelets method [24, 26], Haar wavelets method [7], Chebyshev wavelets method [29, 33, 37], piecewise collocation methods [23, 36], Chebyshev pseudo-spectral method [10, 31], homotopy analysis method [1, 35, 38], homotopy perturbation method [6, 20, 25] and variational iteration method [6, 20].

The main advantage of the sinc-collocation method than other methods is that sinc-collocation method provides a much better rate of convergence and more efficient results in the presence of singularity. For more details about the sinc-collocation method see [2, 3, 4, 34].

Particularly, in the present paper, as an original contribution to literature, sinc-collocation method is introduced for solving linear Volterra-Fredholm integro-differential equations of fractional order. Examined integro-differential equations in the present paper include singularities at some points. Obtained results are given in the form of a new theorem. Some numerical examples in the form of graphs and tables are given to illustrate the theoretical results.

In this study, Volterra-Fredholm integro-differential equations of fractional order are considered as follows:

$$\mu_2(x)y'' + \mu_1(x)y' + \mu_\alpha(x)D_x^\alpha y + \mu_0(x)y = f(x) + \lambda_1 \int_a^x K_1(x,t)y(t)dt + \lambda_2 \int_a^b K_2(x,t)y(t)dt, \quad 0 < \alpha \leq 1 \quad (1.1)$$

in which  $D_x^\alpha$  is the Caputo sense fractional derivative. Eq.(1.1) is subject to following homogeneous boundary conditions

$$y(a) = 0, \quad y(b) = 0, \quad a < x < b.$$

The structure of this paper is organized as follows; In section 2, some preliminaries and basic definitions related to fractional calculus and sinc functions are recalled. In the next section, sinc-collocation method is constructed for solving integro-differential equations of fractional order. In section 4, numerical examples are presented. Finally, conclusions and remarks are given in the section 5.

## 2 Preliminaries

In this section, some preliminaries and notations related to fractional calculus and sinc basis functions are given. For more details see [8, 11, 12, 13, 15, 22, 27, 28, 30].

**Definition 2.1.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function,  $\alpha$  a positive real number,  $n$  the integer satisfying  $n - 1 \leq \alpha < n$ , and  $\Gamma$  the Euler gamma function. Then, the left Caputo fractional derivative of order  $\alpha$  of  $f(x)$  is given as follows:

$$D_x^\alpha f(x) = \frac{1}{\Gamma(n - \alpha)} \int_a^x (x - t)^{n - \alpha - 1} f^{(n)}(t) dt. \quad (2.1)$$

**Definition 2.2.** The Sinc function is defined on the whole real line  $-\infty < x < \infty$  by

$$\text{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0. \end{cases}$$

**Definition 2.3.** For  $h > 0$  and  $k = 0, \pm 1, \pm 2, \dots$  the translated sinc function with space node are given by:

$$S(k, h)(x) = \text{sinc}\left(\frac{x - kh}{h}\right) = \begin{cases} \frac{\sin\left(\pi \frac{x - kh}{h}\right)}{\pi \frac{x - kh}{h}} & x \neq kh \\ 1 & x = kh. \end{cases}$$

To construct approximation on the interval  $(a, b)$  the conformal map

$$\varphi(z) = \ln\left(\frac{z - a}{b - z}\right)$$

is employed. The basis functions on the interval  $(a, b)$  are derived from the composite translated sinc functions

$$S_k(z) = S(k, h)(z) \circ \varphi(z) = \text{sinc}\left(\frac{\varphi(z) - kh}{h}\right).$$

The inverse map of  $w = \varphi(z)$  is

$$z = \varphi^{-1}(w) = \frac{a + be^w}{1 + e^w}.$$

The sinc grid points  $z_k \in (a, b)$  will be denoted by  $x_k$  because they are real. For the evenly spaced nodes  $\{kh\}_{k=-\infty}^{\infty}$  on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \varphi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}}, \quad k = 0, \pm 1, \pm 2, \dots$$

**Definition 2.4.** An open set  $S \subseteq \mathbb{C}$  is called connected if it cannot be written as the union of two disjoint open sets  $A$  and  $B$  such that both  $A$  and  $B$  intersect  $S$ . An open set  $S \subseteq \mathbb{C}$  is called simply connected if  $\overline{\mathbb{C}} \setminus S$ , where  $\overline{\mathbb{C}}$  is the extended complex plane denoted  $\mathbb{C} \cup \{\infty\}$ , is connected.

**Definition 2.5.** Let  $D_E$  be a simply connected domain in the complex plane  $\mathbb{C}$ , and let  $\partial D_E$  denote the boundary of  $D_E$ . Let  $a, b$  be points on  $\partial D_E$  and  $\varphi$  be a conformal map  $D_E$  onto  $D_S$  such that  $\varphi(a) = -\infty$  and  $\varphi(b) = \infty$ . If the inverse map of  $\varphi$  is denoted by  $\varphi$ , define

$$\Gamma = \{\varphi^{-1}(u) \in D_E : -\infty < u < \infty\}$$

and  $z_k = \varphi(kh), k = 0, \pm 1, \pm 2, \dots$

**Definition 2.6.** Let  $B(D_E)$  be the class of functions  $F$  that are analytic in  $D_E$  and satisfy

$$\int_{\psi(L+u)} |F(z)| dz \rightarrow, \text{ as } u = \mp \infty,$$

where

$$L = \left\{ iy : |y| < d \leq \frac{\pi}{2} \right\},$$

and those on the boundary of  $D_E$  satisfy

$$T(F) = \int_{\partial D_E} |F(z)| dz < \infty.$$

**Theorem 2.7.** Let  $\Gamma$  be  $(0, 1), F \in B(D_E)$ , then for  $h > 0$  sufficiently small,

$$\int_{\Gamma} F(z) dz - h \sum_{j=-\infty}^{\infty} \frac{F(z_j)}{\varphi'(z_j)} = \frac{i}{2} \int_{\partial D} \frac{F(z)k(\varphi, h)(z)}{\sin(\pi\varphi(z)/h)} dz \equiv I_F \tag{2.2}$$

where

$$|k(\varphi, h)|_{z \in \partial D} = \left| e^{\left[ \frac{i\pi\varphi(z)}{h} \operatorname{sgn}(\operatorname{Im}\varphi(z)) \right]} \right|_{z \in \partial D} = e^{-\frac{\pi d}{h}}.$$

For the term of fractional in (1.1), the infinite quadrature rule must be truncated to a finite sum. The following theorem indicates the conditions under which an exponential convergence results.

**Theorem 2.8.** If there exist positive constants  $\alpha, \beta$  and  $C$  such that

$$\left| \frac{F(x)}{\varphi'(x)} \right| \leq C \begin{cases} e^{-\alpha|\varphi(x)|} & x \in \psi((-\infty, \infty)) \\ e^{-\beta|\varphi(x)|} & x \in \psi((0, \infty)). \end{cases} \tag{2.3}$$

then the error bound for the quadrature rule (2.3) is

$$\left| \int_{\Gamma} F(x) dx - h \sum_{j=-M}^N \frac{F(x_j)}{\varphi'(x_j)} \right| \leq C \left( \frac{e^{-\alpha Mh}}{\alpha} + \frac{e^{-\beta Nh}}{\beta} \right) + |I_F| \tag{2.4}$$

The infinite sum in (2.3) is truncated with the use of (2.4) to arrive at the inequality (2.5). Making the selections

$$h = \sqrt{\frac{\pi d}{\alpha M}}$$

$$N \equiv \left[ \left[ \frac{\alpha M}{\beta} + 1 \right] \right]$$

where  $[\cdot]$  is an integer part of the statement and  $M$  is the integer value which specifies the grid size, then

$$\int_{\Gamma} F(x) dx = h \sum_{j=-M}^N \frac{F(x_j)}{\varphi'(x_j)} + O\left(e^{-(\pi \alpha d M)^{1/2}}\right). \quad (2.5)$$

We used these theorems to approximate the kernel integral and the arising integral in the formulation of the term fractional in (1.1).

**Lemma 2.9.** Let  $\varphi$  be the conformal one-to-one mapping of the simply connected domain  $D_E$  onto  $D_S$ , given by (2.2). Then

$$\delta_{jk}^{(0)} = [S(j, h) \circ \varphi(x)] \Big|_{x=x_k} \begin{cases} 1 & j = k \\ 0 & j \neq k. \end{cases}$$

$$\delta_{jk}^{(1)} = h \frac{d}{d\varphi} [S(j, h) \circ \varphi(x)] \Big|_{x=x_k} \begin{cases} 0 & j = k \\ \frac{(-1)^{k-j}}{k-j} & j \neq k. \end{cases}$$

$$\delta_{jk}^{(2)} = h^2 \frac{d^2}{d\varphi^2} [S(j, h) \circ \varphi(x)] \Big|_{x=x_k} \begin{cases} -\frac{\pi^2}{3} & j = k \\ \frac{-2(-1)^{k-j}}{(k-j)^2} & j \neq k. \end{cases}$$

### 3 The sinc-collocation method

Let us assume an approximate solution for  $y(x)$  in Eq.(1.1) by finite expansion of sinc basis functions for as follows;

$$y_n(x) = \sum_{k=-M}^N c_k S_k(x), \quad n = M + N + 1 \quad (3.1)$$

where  $S_k(x)$  is the function  $S(k, h) \circ \varphi(x)$ . Here, the unknown coefficients  $c_k$  in (3.1) are determined by sinc-collocation method via the following theorems.

**Theorem 3.1.** The first and second derivatives of  $y_n(x)$  are given by

$$\frac{d}{dx} y_n(x) = \sum_{k=-M}^N c_k \varphi'(x) \frac{d}{d\varphi} S_k(x) \quad (3.2)$$

$$\frac{d^2}{dx^2} y_n(x) = \sum_{k=-M}^N c_k \left( \varphi''(x) \frac{d}{d\varphi} S_k(x) + (\varphi')^2 \frac{d^2}{d\varphi^2} S_k(x) \right) \quad (3.3)$$

respectively.

**Theorem 3.2.** If  $\xi$  is a conformal map for the interval  $[a, x]$ , then  $\alpha$  order derivative of  $y_n(x)$  for  $0 < \alpha < 1$  is given by

$$D_x^\alpha(y_n(x)) = \sum_{k=-M}^N c_k D_x^\alpha(S_k(x)) \tag{3.4}$$

where

$$D_x^\alpha(S_k(x)) \approx \frac{h_L}{\Gamma(1-\alpha)} \sum_{r=-L}^L \frac{(x-x_r)S'_k(x_r)}{\xi'(x_r)}$$

*Proof.* If we use the definition of Caputo fractional derivative given in (2.1), it is written that

$$D_x^\alpha(y_n(x)) = \sum_{k=-M}^N c_k D_x^\alpha(S_k(x))$$

where

$$D_x^\alpha(S_k(x)) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} S'_k(t) dt$$

Now we use quadrature rule given by (2.5) to compute the above integral which is divergent on the interval  $[a, x]$ . For this purpose, a conformal map and its inverse image that denotes the sinc grid points are given by

$$\xi(t) = \ln\left(\frac{t-a}{x-t}\right)$$

and

$$x_r = \xi^{-1}(rh_L) = \frac{a + xe^{rh_L}}{1 + e^{rh_L}}$$

where  $h_L = \pi/\sqrt{L}$ . Then, according to equality (2.5), we can write

$$D_x^\alpha(S_k(x)) \approx \frac{h_L}{\Gamma(1-\alpha)} \sum_{r=-L}^L \frac{(x-x_r)S'_k(x_r)}{\xi'(x_r)}$$

This completes the proof.

Q.E.D.

Application of equality (2.5) to the kernel integral in (1.1) gives the following two lemmas.

**Lemma 3.3.** The following relation holds

$$\int_a^{x_j} K_1(x,t)y(t)dt \approx h \sum_{k=-M}^N \delta_{jk}^{(-1)} \frac{K_1(x_j, t_k)}{\varphi'(t_k)} y_k \tag{3.5}$$

where

$$\sigma_{jk} = \int_0^{j-k} \frac{\sin \pi t}{\pi t} dt$$

$$\delta_{jk}^{(-1)} = \frac{1}{2} + \sigma_{jk}$$

and  $y_k$  denotes an approximate value of  $y(t_k)$ .

**Lemma 3.4.** The following relation holds

$$\int_a^b K_2(x, t)y(t)dt \approx h \sum_{k=-M}^N \frac{K_2(x, t_k)}{\varphi'(t_k)} y_k \quad (3.6)$$

where  $y_k$  denotes an approximate value of  $y(t_k)$ .

Replacing each term of (1.1) with the approximation given in (3.1)-(3.6), we obtain the following system

$$\sum_{k=-M}^N \left[ c_k \left\{ \mu_2(x) \left( \varphi''(x) \frac{d}{d\varphi} S_k(x) + (\varphi'(x))^2 \frac{d^2}{d\varphi^2} S_k(x) \right) + \mu_1(x) \varphi'(x) \frac{d}{d\varphi} S_k(x) + \mu_\alpha(x) D_x^\alpha(S_k(x)) \right. \right. \\ \left. \left. + \mu_0(x) S_k(x) - \lambda_1 h \delta_{jk}^{(-1)} \frac{K_1(x, t_k)}{\varphi'(t_k)} - \lambda_2 h \frac{K_2(x, t_k)}{\varphi'(t_k)} \right\} \right] = f(x)$$

Then, multiplying the resulting equation by  $\{(1/\varphi'(x))^2\}$ , we obtain

$$\sum_{k=-M}^N \left[ c_k \left\{ \mu_2(x) \frac{d^2}{d\varphi^2} S_k(x) + \left( \mu_2(x) \frac{\varphi''(x)}{(\varphi'(x))^2} + \mu_1(x) \frac{1}{\varphi'(x)} \right) \frac{d}{d\varphi} S_k(x) + \mu_\alpha(x) \left( \frac{1}{\varphi'(x)} \right)^2 D_x^\alpha(S_k(x)) \right. \right. \\ \left. \left. + \mu_0(x) \left( \frac{1}{\varphi'(x)} \right)^2 S_k(x) - \lambda_1 h \delta_{jk}^{(-1)} \frac{K_1(x, t_k)}{\varphi'(x)^2 \varphi'(t_k)} - \lambda_2 h \frac{K_2(x, t_k)}{\varphi'(x)^2 \varphi'(t_k)} \right\} \right] = f(x) \left( \frac{1}{\varphi'(x)} \right)^2$$

Here, by using the following equality

$$\frac{\varphi''(x)}{(\varphi')^2} = - \left( \frac{1}{\varphi'(x)} \right)'$$

we can write

$$\sum_{k=-M}^N \left[ c_k \left\{ \sum_{i=0}^2 g_i(x) \frac{d^i}{d\varphi^i} S_k + g_3(x) D_x^\alpha(S_k(x)) + g_4(x) \delta_{jk}^{(-1)} \frac{K_1(x, t_k)}{\varphi'(t_k)} + g_5(x) \frac{K_2(x, t_k)}{\varphi'(t_k)} \right\} \right] \\ = \left( f(x) \left( \frac{1}{\varphi'(x)} \right)^2 \right)$$

where

$$\begin{aligned} g_0(x) &= \mu_0(x) \left( \frac{1}{\varphi'(x)} \right)^2 \\ g_1(x) &= \left[ \mu_1(x) \left( \frac{1}{\varphi'(x)} \right) - \mu_2(x) \left( \frac{1}{\varphi'(x)} \right)' \right] \\ g_2(x) &= \mu_2(x) \\ g_3(x) &= \mu_\alpha(x) \left( \frac{1}{\varphi'(x)} \right)^2 \\ g_4(x) &= -\lambda_1 h \left( \frac{1}{\varphi'(x)} \right)^2 \\ g_5(x) &= -\lambda_2 h \left( \frac{1}{\varphi'(x)} \right)^2 \end{aligned}$$

It's known from [18] that

$$\delta_{jk}^{(0)} = \delta_{kj}^{(0)}, \quad \delta_{jk}^{(1)} = -\delta_{kj}^{(1)}, \quad \delta_{jk}^{(2)} = \delta_{kj}^{(2)}$$

So, we obtain the following theorem.

**Theorem 3.5.** If the assumed approximate solution of boundary value problem (1.1) is (3.1), then the discrete sinc-collocation system for the determination of the unknown coefficients  $\{c_k\}_{k=-M}^N$  is given by

$$\begin{aligned} \sum_{k=-M}^N \left[ c_k \left\{ \sum_{i=0}^2 \frac{g_i(x_j)}{h^i} \delta_{jk}^{(i)} + g_3(x_j) D_x^\alpha(S_k(x_j)) + g_4(x_j) \delta_{jk}^{(-1)} \frac{K_1(x_j, t_k)}{\varphi'(t_k)} + g_5(x_j) \frac{K_2(x_j, t_k)}{\varphi'(t_k)} \right\} \right] \\ = \left( f(x_j) \left( \frac{1}{\varphi'(x_j)} \right)^2 \right), \quad j = -M, \dots, N \end{aligned} \tag{3.7}$$

We now introduce some notations to rewrite in the matrix form for system (3.7). Let  $\mathbf{D}(y)$  denotes a diagonal matrix whose diagonal elements are  $y(x_{-M}), y(x_{-M+1}), \dots, y(x_N)$  and non-diagonal elements are zero, let

$$\begin{aligned} \mathbf{G} &= D_x^\alpha(S_k(x_j)) \\ \mathbf{E}_1 &= \frac{K_1(x_j, t_k)}{(\varphi'(x_j))^2 \varphi'(t_k)} \end{aligned}$$

and

$$\mathbf{E}_2 = \frac{K_2(x_j, t_k)}{(\varphi'(x_j))^2 \varphi'(t_k)}$$

denote a matrix and also let  $\mathbf{I}^{(i)}$  denote the matrices

$$\mathbf{I}^{(i)} = [\delta_{jk}^{(i)}], \quad i = -1, 0, 1, 2$$

where  $\mathbf{D}, \mathbf{G}, \mathbf{E}_1, \mathbf{E}_2, \mathbf{I}^{(-1)}, \mathbf{I}^{(0)}, \mathbf{I}^{(1)}$  and  $\mathbf{I}^{(2)}$  are square matrices of order  $n \times n$ . In order to calculate unknown coefficients  $c_k$  in linear system (3.7), we rewrite this system by using the above notations in matrix form as

$$\mathbf{A} \mathbf{c} = \mathbf{B} \tag{3.8}$$

where

$$\begin{aligned} \mathbf{A} &= \sum_{i=0}^2 \frac{1}{h^i} \mathbf{D}(g_i) \mathbf{I}^{(i)} + \mathbf{D}(g_3) \mathbf{G} + \mathbf{D}(g_4) (\mathbf{E}_1 \circ \mathbf{I}^{(-1)}) + \mathbf{D}(g_5) \mathbf{E}_2 \\ \mathbf{B} &= \mathbf{D} \left( \frac{f}{(\varphi')^2} \right) \mathbf{1} \\ \mathbf{c} &= (c_{-M}, c_{-M+1}, \dots, c_N)^T \end{aligned}$$

The notation "  $\circ$  " denotes the Hadamard matrix multiplication. Now we have linear system of  $n$  equations in the  $n$  unknown coefficients given by (3.8). We can find the unknown coefficients  $c_k$  by solving this system.

Table 1: Absolute errors for *Example 1* for  $N = 32$  and different values of  $\alpha$ 

$x$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$
0.1	$3.395 \times 10^{-3}$	$1.945 \times 10^{-3}$	$1.181 \times 10^{-3}$
0.2	$6.465 \times 10^{-3}$	$3.602 \times 10^{-3}$	$2.170 \times 10^{-3}$
0.3	$8.649 \times 10^{-3}$	$4.551 \times 10^{-3}$	$2.774 \times 10^{-3}$
0.4	$9.362 \times 10^{-3}$	$4.411 \times 10^{-3}$	$2.814 \times 10^{-3}$
0.5	$8.227 \times 10^{-3}$	$3.016 \times 10^{-3}$	$2.205 \times 10^{-3}$
0.6	$5.285 \times 10^{-3}$	$5.421 \times 10^{-4}$	$1.018 \times 10^{-3}$
0.7	$1.175 \times 10^{-3}$	$2.394 \times 10^{-3}$	$4.705 \times 10^{-4}$
0.8	$2.719 \times 10^{-3}$	$4.665 \times 10^{-3}$	$1.728 \times 10^{-3}$
0.9	$4.167 \times 10^{-3}$	$4.607 \times 10^{-3}$	$1.947 \times 10^{-3}$

#### 4 Computational examples

In this section, two problems that have homogeneous boundary conditions will be tested by using the present method via *Mathematica10*. In the first example a problem that has the known exact solution for integer order derivative case is considered. So one could compare the obtained results from the proposed numerical algorithm with the exact solution. Then the second example is given to show the efficiency of the proposed method for the singular problems. In the both examples, we take  $h = \pi/\sqrt{N}$ ,  $L = N = M$ .

**Example 1.** Let us first consider the linear fractional integro-differential equation

$$y''(x) + D_x^\alpha y(x) = f(x) - 2 \int_0^x K_1(x, t)y(t)dt + \int_0^1 K_2(x, t)y(t)dt$$

subject to the homogeneous boundary conditions

$$y(0) = 0, \quad y(1) = 0$$

where  $f(x) = -\frac{1}{30} - 6x + \frac{181x^2}{20} + 4x^3 - \frac{x^5}{10} + \frac{x^6}{15}$ ,  $K_1(x, t) = x - t$  and  $K_2(x, t) = x^2 - t$ . The exact solution of this problem for  $\alpha = 1$  is  $y(x) = x^3(x - 1)$ . The numerical solutions which are obtained by using the present method for  $N = 32$  and different values of  $\alpha$  are presented in Table 1. Also, the graphs of approximate solutions for different values of  $\alpha$  are given in Figure 1. Graphs in Figure 1 show that when  $\alpha$  approaches to 1, the corresponding solutions of fractional order differential equation approach to the solutions of integer order differential equation.

**Example 2.** Consider the linear singular fractional integro-differential equation

$$y''(x) + \frac{1}{x}D_x^{0.5}y(x) + \frac{1}{x^2}y(x) = f(x) + \int_0^x K_1(x, t)y(t)dt + \int_0^1 K_2(x, t)y(t)dt$$

subject to the homogeneous boundary conditions

$$y(0) = 0, \quad y(1) = 0$$



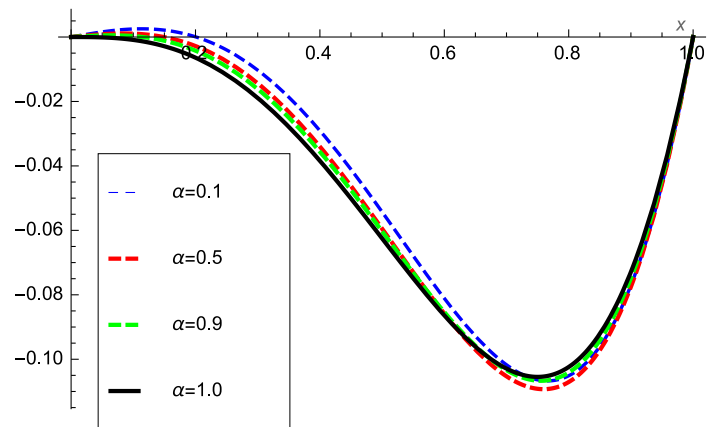


Figure 1: Graphs of approximate solutions for *Example 1* for  $N = 32$  and different values of  $\alpha$

where  $f(x) = 5 + 1.50451x^{0.5} - 13x - 1.80541x^{1.5} - x^2 + x^3 - 2.0674 \cos(x) + 5.95385 \sin(x)$ ,  $K_1(x, t) = \sin(x - t)$  and  $K_2(x, t) = \cos(x - t)$ . The exact solution of this problem is  $y(x) = x^2(1 - x)$ . The numerical solutions which are obtained by using the present method for different values of  $N$  are presented in Table 2. Also, the graphs of approximate solutions for different values of  $N$  are given in Figure 2.

Table 2: Absolute errors for *Example 2* for different values of  $N$ 

$x$	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
0.1	$6.570 \times 10^{-3}$	$1.160 \times 10^{-2}$	$6.951 \times 10^{-4}$	$7.309 \times 10^{-6}$	$6.417 \times 10^{-8}$
0.2	$2.330 \times 10^{-2}$	$2.277 \times 10^{-2}$	$1.546 \times 10^{-3}$	$2.048 \times 10^{-5}$	$2.931 \times 10^{-7}$
0.3	$2.989 \times 10^{-2}$	$2.678 \times 10^{-2}$	$1.755 \times 10^{-3}$	$2.606 \times 10^{-5}$	$3.853 \times 10^{-7}$
0.4	$2.690 \times 10^{-2}$	$2.572 \times 10^{-2}$	$1.632 \times 10^{-3}$	$2.503 \times 10^{-5}$	$3.915 \times 10^{-7}$
0.5	$1.876 \times 10^{-2}$	$2.158 \times 10^{-2}$	$1.456 \times 10^{-3}$	$2.221 \times 10^{-5}$	$3.449 \times 10^{-7}$
0.6	$1.011 \times 10^{-2}$	$1.656 \times 10^{-2}$	$1.209 \times 10^{-3}$	$1.789 \times 10^{-5}$	$2.696 \times 10^{-7}$
0.7	$4.615 \times 10^{-3}$	$1.223 \times 10^{-2}$	$8.495 \times 10^{-4}$	$1.202 \times 10^{-5}$	$1.838 \times 10^{-7}$
0.8	$3.697 \times 10^{-3}$	$8.652 \times 10^{-3}$	$5.251 \times 10^{-4}$	$7.682 \times 10^{-6}$	$1.011 \times 10^{-7}$
0.9	$4.256 \times 10^{-3}$	$4.091 \times 10^{-3}$	$2.684 \times 10^{-4}$	$3.034 \times 10^{-6}$	$3.596 \times 10^{-8}$

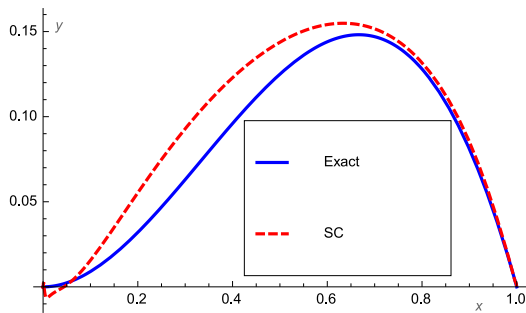
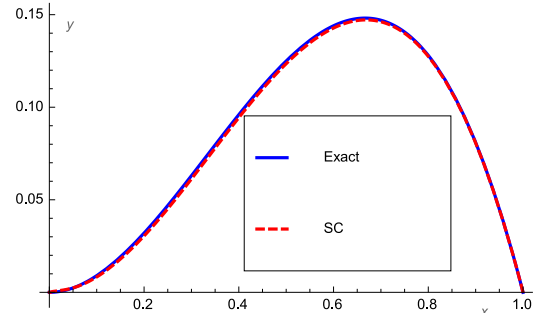
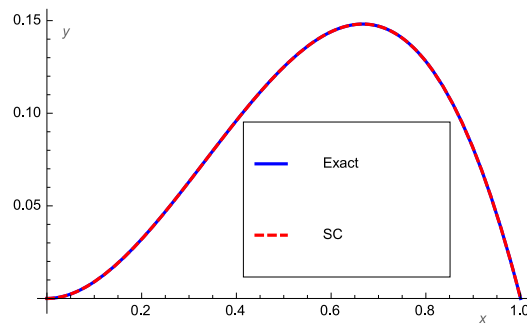
(a)  $N = 4$ (b)  $N = 16$ (c)  $N = 64$ 

Figure 2: Graphs of exact and approximate solutions for Example 2

## 5 Conclusion

In this paper, sinc-collocation method is used to solve a class of fractional Volterra-Fredholm integro-differential equation. In order to illustrate the accuracy and effective of the method, it is applied to some examples and obtained results are compared with the exact ones. The comparisons in table and graphical forms show that the approximate solutions converge the exact ones when it is increased that the number of sinc grid points  $N$  and the present method is a powerful tool for solving fractional integro-differential equations with boundary conditions.

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